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Computing topological invariants with one and two-matrix models

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Abstract: A generalization of the Kontsevich Airy-model allows one to compute the intersection numbers of the moduli space of p-spin curves. These models are deduced from averages of characteristic polynomials over Gaussian ensembles of random matrices in an external matrix source. After use of a duality, and of an appropriate tuning of the source, we obtain in a double scaling limit these intersection numbers as polynomials in p. One can then take the limit $p \to -1$ which yields a matrix model for orbifold Euler characteristics. The generalization to a time-dependent matrix model, which is equivalent to a two-matrix model, may be treated along the same lines; it also yields a logarithmic potential with additional vertices for general p.

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Contents

1 **Introduction** 1

2 **Replica and duality for the one-matrix model** 2

3 **The p-dependence of the intersection numbers with one marked point** 7

4 **The n-point correlation functions** 11

5 **Time-dependent Gaussian matrix model** 14

6 **Duality formula for the two-matrix model** 15

7 **Discussion** 21

A **Three point correlation function $\mathrm{U}(s_1, s_2, s_3)$** 22

B **The n-point correlation function for $n \geq 4$** 24

C **Ginzburg-Landau potential for primary fields and Gelfand-Dikii equation** 26

1 Introduction

Many topological invariants have been computed from matrix models of moduli spaces. The well-known Kontsevich's Airy matrix model [1] gives the intersection numbers of the moduli spaces of curves, which was also studied by the double scaling limit of one matrix model [2]. The Euler characteristics of orbifolds have been computed from the Penner model [3]. The intersection numbers for the p-spin curves are obtained from the generalized Kontsevich model [4]. These matrix models all provide explicit results for the intersection numbers.

In this article, we discuss the above models, which lead to three kinds of matrix models, in a unified way. Our formulation starts from simple Gaussian matrix models with an external matrix source. In recent articles we have already considered the average characteristic polynomials in these Gaussian ensembles, and derived, through a duality relation and the replica method, the intersection numbers of p-spin curves [5–7]. This duality relates the average of the product of k characteristic polynomials for $N \times N$ random matrices M, to the average of the product of N characteristic polynomials over $k \times k$ Gaussian random matrices B. In the large N limit, the matrix model for B reduces to the higher Airy matrix [6, 7] for the intersection numbers of spin curves studied by Witten [4, 8].

This duality allows one to compute the intersection numbers for the spin moduli spaces with n-marked points and genus g, from an n-point correlation function $\mathrm{U}(s_1, \ldots, s_n)$ of

Gaussian random matrices in a scaling limit near critical edges [9, 10]. The basic steps are recalled in section 2.

In this article we first compute explicitly the intersection numbers of moduli space of p-spin curves with one marked point, for arbitrary values of p, as polynomials in p. We have obtained earlier the intersection numbers for p=2,3 and 4 explicitly, but we discuss here arbitrary p. This allows us to consider continuations in p; in particular the limit $p \to -1$ exhibits an interesting relation between the intersection numbers, (τ-class) and the orbifold Euler characteristics $\chi(M_{g,1}) = \zeta(1-2g)$ ($\zeta(x)$ is Riemann zeta function) [3, 11]. In section 3 we derive these numbers for surfaces with one marked point.

In section 4 we show that the intersection numbers with n-marked points for p-spin curves are also obtained easily from the integral representation of $\mathrm{U}(s_1, \ldots, s_n)$ at the critical values tuned through an appropriate external source. We evaluate the case of two marked points for genus one (g=1) and any p. The generating function U is given for three and four marked points in appendices A and B. Our results through this generating function U are consistent with the previous recursive results of Virasoro equations. We find that the ring structure of the primary fields, for genus zero, is deduced from these n-point correlation functions U for arbitrary p. This shows that the random matrix theory with external source near critical edges, has a structure of a minimal $N = 2$ superconformal field theory with Lie algebra A_{p-1}.

It has been conjectured by Witten that the free energy F which generates the intersection numbers of the moduli space of p-spin curves satisfies a Gelfand-Dikii hierarchy [4, 12]. We show here that the intersection numbers, computed from the integral representation of $\mathrm{U}(s_1, \ldots, s_n)$, do satisfy Gelfand-Dikii equations. We present in appendix C, this Gelfand-Dikii hierarchy equations and the construction of the super potential for the primary fields. We also note that, with respect to Witten's conjecture, that the definition of intersection numbers as an integral over the compactified moduli space $\bar{M}_{g,n}$, is similar in structure to the integral representation of $\mathrm{U}(s_1, \ldots, s_n)$.

In section 5, begins a second part of the paper devoted to the time dependent Gaussian matrix model for which we extend our previous work on duality [5–7]. The time-dependent model, a matrix quantum mechanics of harmonic oscillators, reduces easily, for a Gaussian distribution, to an equivalent two-matrix model. Again one may derive (section 6) a dual model in the presence of a matrix source. We then obtain, with an appropriate tuning of the source, the two-matrix equivalents of the Kontsevich plus Penner models for matter with central charge $c = 1$ for the $p = 2$ case. For $p > 2$, additional terms are present with respect to the c=1 matrix model for tachyon correlators [14, 15].

2 Replica and duality for the one-matrix model

Let us first summarize the steps which one uses to obtain Airy and higher Airy matrix models from the Gaussian one-matrix model in a source, followed by the computation of the intersection numbers through the replica method [5–7].

The m-point correlation functions of the eigenvalues in the Gaussian unitary ensemble are conveniently deduced from their Fourier transforms $\mathrm{U}(s_1,\ldots,s_m)$, defined as

$$\mathrm{U}(s_1,s_2,\ldots,s_m) = <\mathrm{tr}e^{s_1M}\mathrm{tr}e^{s_2M}\cdots\mathrm{tr}e^{s_mM}> \\ = \int\prod_{l=1}^{m}d\lambda_l e^{\sum it_l\lambda_l}<\prod_{1}^{m}\mathrm{tr}\delta(\lambda_j-M)> \tag{2.1}$$

where $s_l = it_l$; M is an $N\times N$ Hermitian random matrix. The bracket stands for averages with the Gaussian probability measure

$$<X>_A = \int dMe^{-\frac{N}{2}\mathrm{tr}M^2+N\mathrm{tr}MA}X(M), \tag{2.2}$$

A is an $N\times N$ external Hermitian source matrix. We may assume that this matrix A is diagonal with eigenvalues a_j. We consider later the external source A with $(p-1)$ distinct eigenvalues, each of them being $\frac{N}{p-1}$ times degenerate. This parameter p is crucial in this paper.

Let us consider first the one point case (m=1), namely $\mathrm{U}(s)$. The replica limit $k\to 0$ for $<\mathrm{tr}\delta(\lambda-M)>$ relies on the identity

$$\lim_{k\to 0}\frac{1}{k}\left([\det(\lambda\cdot\mathrm{I}-M)]^k-1\right) = \mathrm{trlog}(\lambda\cdot\mathrm{I}-M) \tag{2.3}$$

Taking a derivative with respect to λ, it yields the density of states $\mathrm{tr}\delta(\lambda-M)$ as the imaginary part of the resolvent when the imaginary part of λ goes to zero. Thus $\mathrm{U}(s)$ is expressed in terms of products of characteristic polynomials,

$$\mathrm{U}(s) = \lim_{k\to 0}\frac{1}{k}\int d\lambda e^{s\lambda}\frac{\partial}{\partial\lambda}<\prod_{\alpha=1}^{k}\det(\lambda_\alpha-M)>_A|_{\lambda_\alpha=\lambda} \tag{2.4}$$

We have introduced a replica symmetry breaking by taking k distinct λ_α $(\alpha=1,\ldots,k)$ [16], in order to use the duality formula [6, 17],

$$\frac{1}{Z_0}<\prod_{\alpha=1}^{k}\det(\lambda_\alpha-M)>_A = \frac{1}{Z_0'}<\prod_{j=1}^{N}\det(a_j-iB)>_\Lambda . \tag{2.5}$$

where $\Lambda = diag(\lambda_1,\ldots,\lambda_n)$, B is an $k\times k$ Hermitian random matrix, and Z_0 and Z_0' are normalization constants. The probability distribution for B in the right hand side is

$$<Y>_\Lambda = \int dBe^{-\frac{N}{2}\mathrm{tr}B^2+iN\mathrm{tr}B\Lambda}Y(B) \tag{2.6}$$

The simplest case consists of taking an external source multiple of the identity, namely $a_j = 1,\ j=1,\ldots N$. The effect of this source is simply to shift all the eigenvalues of M by one: the left edge of Wigner's semi-circle is now at the origin. In the large N limit, after exponentiation, expansion of the integrand in powers of B, cancellation of the trB^2 terms, we obtain

$$Z = \lim_{N\to\infty}<[\det(1-iB)]^N>_\Lambda = \int dBe^{\frac{iN}{3}\mathrm{tr}B^3+iN\mathrm{tr}B(\Lambda-1)} \tag{2.7}$$

which is Kontsevich Airy matrix-model. (We explore here a "double scaling limit", namely the vicinity of the origin, in which the $(\lambda_a - 1)$ are of order $N^{-2/3}$, B of order $N^{-1/3}$; in this regime $N \mathrm{tr} B^l$ is negligible for $l \geq 4$). This Airy matrix model has an expansion in terms of the moduli parameters t_m,

$$t_m = C \mathrm{tr} \frac{1}{\Lambda^{2m+1}} = C \sum_{\alpha=1}^{k} \frac{1}{\lambda_\alpha^{2m+1}} \tag{2.8}$$

$$Z = \sum_{m,k_m} < \prod_m \tau_m^{k_m} > \prod_m \frac{t_m^{k_m}}{k_m!} \tag{2.9}$$

where C is a normalization constant, to be determined later. When we set the k distinct λ_α to a common value λ, we have $t_m = C \frac{k}{\lambda^{2m+1}}$. Then, in the limit $k \to 0$, only single τ's appear, since the parameter t_m is proportional to k. Thus the zero-replica limit $k \to 0$ yields the intersection numbers with one-marked point (one t_m):

$$\mathrm{U}(s) = \int d\lambda e^{s\lambda} \frac{\partial}{\partial \lambda} [1 + < \tau_1 > t_1 + < \tau_4 > t_4 + \cdots] \tag{2.10}$$

where $< \tau_1 >= \frac{1}{24}$, $< \tau_4 >= \frac{1}{(24)^2 2!}$. As found in [5], $\mathrm{U}(s)$ is obtained, after approriate normalization, as

$$\mathrm{U}(s) = \frac{1}{s^{3/2}} e^{\frac{s^3}{24}} \tag{2.11}$$

which gives the intersection number $< \tau_m >$ for the moduli space of curves with one marked point,

$$< \tau_m >= \frac{1}{(24)^g g!} \tag{2.12}$$

where g is the genus of the curve and $m = 3g - 2$. We have thus shown that the Fourier transform $\mathrm{U}(s)$ gives the intersection numbers $< \tau_m >$ of the moduli space of curves with one marked point [5, 18].

The replica limit $k \to 0$, where the matrix B is $k \times k$, was studied in [6], and it gives the intersection numbers of (2.12). Note that in the original Kontsevich model of (2.7), the matrix size k was arbitrary, and the universal intersection numbers $< \tau_m >$ are independent of k.

From other tunings of the external source a_j, we may obtain also the intersection numbers of the moduli space of p-spin curves [4] with one marked point, which exhibit "spin structures". Indeed we may tune the external source so that the asymptotic density of states vanishes at an edge as $\rho(\lambda) \sim \lambda^{\frac{1}{p}}$. This will yield the exact values for p-spin curves with genus g and one marked point. A spin index $j = 0, 1, \ldots, p-1$ is now needed. From this tuning of the external source, we obtain the generalized Kontsevich model,

$$Z = \int dB e^{\frac{i}{p+1} \mathrm{tr} B^{p+1} - i \mathrm{tr} \mathrm{B} \Lambda^{\mathrm{p}}} \tag{2.13}$$

where B is $k \times k$. The derivation of this partition function from the right hand side of the duality formula (2.5) will be given in the next section. This Z has an expansion,

$$Z = \sum_{m,j,k_{m,j}} < \prod_{m,j} \tau_{m,j}^{k_{m,j}} > \prod_m \frac{t_{m,j}^{k_{m,j}}}{k_{m,j}!} \tag{2.14}$$

JHEP04(2009)110

where

$$t_{m,j} = (-p)^{\frac{j-p-m(p+2)}{2(p+1)}} \prod_{l=0}^{m-1}(1+j+lp)\mathrm{tr}\frac{1}{\Lambda^{pm+j+1}}. \tag{2.15}$$

The normalization constant C in (2.8) is fixed by (2.15). The intersection numbers of moduli space of p-spin curves are defined by the integral formula of compactified the moduli space $\bar{M}_{g,n}$ [8]

$$< \tau_{n_1}(U_{j_1})\cdots\tau_{n_s}(U_{j_s}) >= \frac{1}{(\hat{k}+2)^g}\int_{\bar{M}_{g,s}} C_T(\nu)\prod_{i=1}^{s}(c_1(\mathcal{L}_i))^{n_i} \tag{2.16}$$

where U_j is an operator for the primary matter field (tachyon), related to top Chern class $C_T(\nu)$, and τ_n is a gravitational operator, related to the first Chern class c_1 of the line bundle $\mathcal{L}_i$ at the ith-marked point. We denote $\tau_n(U_j)$ by $\tau_{n,j}$, and j represents the spin index (j=0,…,p-1). The indices n_i and j_i are related to genus g and numbers of marked points s through

$$(p+1)(2g-2+s) = \sum_{i=1}^{s}(pn_i + j_i + 1). \tag{2.17}$$

This intersection theory for spin-curves [4] is known to be related to the minimal $N = 2$ superconformal field theory of Lie algebra A_{p-1} type, which is equivalent to $\mathrm{SU}(2)_{\hat{k}}/\mathrm{U}(1)$ Wess-Zumino-Witten model. $\hat{k}$ is a number of levels and it is related to p by $\hat{k} = p-2$. This relation is derived from a super potential W for the chiral ring structures of primary fields; we will obtain this chiral structure later by the consideration of the n-point correlation functions $\mathrm{U}(s_1,\ldots,s_n)$ (see appendix C).

As remarked by Witten [4], the limit $p \to -1$ ($\hat{k} \to -3$) corresponds to the top Chern class without gravity decendants $c_1(\mathcal{L}_i)$, and this top Chern class becomes the orbifold Euler characteristic class [3, 11].

For $3 \leq p$, we have to consider the above spin structures for the intersection numbers. We find the intersection number with one marked point for arbitrary genus, $< \tau_{n,j} >$ [7] for p=3 as

$$< \tau_{n,j} >_g = \frac{1}{(12)^g g!}\frac{\Gamma\left(\frac{g+1}{3}\right)}{\Gamma\left(\frac{2-j}{3}\right)} \tag{2.18}$$

where $n = (8g-5-j)/3$ and $j = 0$ for $g = 3m+1$, $j = 1$ for $g = 3m$ ($m = 1, 2, \ldots$).

In the replica limit, $k \to 0$ for the matrix B, a closed expression for $U_0(s_1,\ldots,s_k)$ is known [6] (the surfix "0" refers to zero external source in the Gaussian probability distribution),

$$\lim_{k\to 0} U_0(s_1,\ldots,s_n) = \frac{1}{\sigma^2}\prod_{i=1}^{n} 2\mathrm{sinh}\left(\frac{s_i\sigma}{2}\right) \tag{2.19}$$

where $\sigma = \sum_{i=1}^{n} s_i$, in which $U_0(s_1,\ldots,s_n)$ is defined by

$$\begin{aligned} U_0(s_1,\ldots,s_n) &= < \mathrm{tr}e^{s_1 B}\cdots\mathrm{tr}e^{s_n B} > \\ &= \sum_{l_i} < \prod_{i=1}^{n}\mathrm{tr}B^{l_i} > \prod_{i=1}^{n}\frac{s_i^{l_i}}{l_i!}. \end{aligned} \tag{2.20}$$

JHEP04(2009)110

From (2.19), we obtain the intersection numbers of p-spin curves with one marked point. The replica limit $k \to 0$ selects only the one marked point ribbon graphs on a genus g-Riemann surface. This method gives the intersection numbers of p-spin curves with one marked point, and the results coincide with those obtained from $\mathrm{U}(s)$ [6].

For two marked points, one deals with the dual quantity $\mathrm{U}(s_1, s_2)$, again at a critical edge point. The correspondence is the same as for the one-marked point. We have

$$
\begin{aligned}
\mathrm{U}(s_1, s_2) &= < \mathrm{tr} e^{s_1 M} \mathrm{tr} e^{s_2 M} > \\
&= \int d\lambda_1 d\lambda_2 e^{s_1\lambda_1 + s_2\lambda_2} < \mathrm{tr}\delta(\lambda_1 - M)\mathrm{tr}\delta(\lambda_2 - M) > \\
&= \lim_{k_1,k_2\to 0} \int d\lambda_1 d\lambda_2 e^{s_1\lambda_1 + s_2\lambda_2} \frac{\partial^2}{\partial\lambda_1\partial\lambda_2} < [\det(\lambda_1 - M)]^{k_1} [\det(\lambda_2 - M)]^{k_2} > \\
&= \lim_{k_1,k_2\to 0} \int d\lambda_1 d\lambda_2 e^{s_1\lambda_1 + s_2\lambda_2} \frac{\partial^2}{\partial\lambda_1\partial\lambda_2} < [\det(1 - iB)]^N >_\Lambda
\end{aligned} \tag{2.21}
$$

where $\Lambda = diag(\lambda_1, \ldots, \lambda_1, \lambda, \ldots, \lambda_2)$, λ_1 and λ_2 are degenerate k_1 and k_2 times respectively. The matrix B is an $(k_1 + k_2) \times (k_1 + k_2)$ Hermitian matrix. In the limit of zero replica, one selects the terms of order $k_1 k_2$ in the Airy matrix model, for instance a term like $\frac{k_1}{\lambda_1^3}\frac{k_2}{\lambda_2^3}$, and we obtain the two marked points contribution for the intersection numbers. The Fourier transform $\mathrm{U}(s_1, s_2)$, with respect to λ_1 and λ_2, gives the intersection numbers as coefficients of the Taylor expansion in s_1 and s_2. For the case p=2, this was checked for arbitrary genus [5] and it does yield the known values.

For higher marked points, the argument is similar. For n marked points, one considers the terms

$$
\sum < \tau_{m_1} \cdots \tau_{m_n} > \frac{k_1 \cdots k_n}{\lambda_1^{2m_1+1} \lambda_2^{2m_2+1} \cdots \lambda_n^{2m_n+1}} \tag{2.22}
$$

emerging from the B-matrix integral. The Fourier transform of these quantities is given by $\mathrm{U}(s_1, s_2, \ldots, s_n)$.

Thus we have indeed a method for computing the intersection numbers of the moduli of curves from random matrix theory, based on the expression for $\mathrm{U}(s_1, \ldots, s_n)$.

An exact and useful integral representation for $\mathrm{U}(s_1, \ldots, s_n)$ is known in the presence of an external matrix source A [19].

$$
\begin{aligned}
\mathrm{U}(s_1, \ldots, s_n) &= \frac{1}{N} \langle \mathrm{tr e^{s_1 M}} \cdots \mathrm{tr e^{s_n M}} \rangle \\
&= e^{\sum_1^n s_i^2} \oint \prod_1^n \frac{du_i}{2\pi i} e^{\sum_1^n u_i s_i} \prod_{\alpha=1}^N \prod_{i=1}^n \left(1 - \frac{s_i}{a_\alpha - u_i}\right) \det \frac{1}{u_i - u_j + s_i}
\end{aligned} \tag{2.23}
$$

We will use this formula in the following sections to obtain the intersection numbers of p-spin curves of arbitrary genus g for n-marked points, through an appropriate tuning of the external source a_j, in a scaling large N limit.

JHEP04(2009)110

3 The p-dependence of the intersection numbers with one marked point

The partition function Z_p for the generalized Kontsevich model is given by the $k \times k$ Hermitian matrix B,

$$Z_p = \int dB e^{\frac{1}{p+1}\mathrm{tr}B^{p+1} - \mathrm{tr}B\Lambda} \tag{3.1}$$

This model is obtained, after use of the duality, from the expectation values of characteristic polynomials (2.5). We take here an external source A with $(p-1)$ distinct eigenvalues, each of them being $\frac{N}{p-1}$ times degenerate: $A = \mathrm{diag}(a_1, \ldots, a_1, \ldots, a_{p=1}, \ldots, a_{p-1})$. After duality, the expectation values of characteristic polynomials become

$$< \prod_{\alpha=1}^{p-1} \det(a_\alpha - iB)^{\frac{N}{p-1}} >=< \exp\left[\sum_{\alpha=1}^{p-1} \mathrm{trlog}\left(1 - \frac{iB}{a_\alpha}\right) + N \sum \log\left(\prod_{\alpha=1}^{p-1} a_\alpha\right)\right] > \tag{3.2}$$

We now specify the $(p-1)$ distinct eigenvalues of the external source by the $(p-1)$ conditions:

$$\sum_{\alpha}^{p-1} \frac{1}{a_\alpha^2} = p-1, \qquad \sum_{\alpha=1}^{p-1} \frac{1}{a_\alpha^m} = 0, \qquad (m = 3, 4, \ldots, p)$$
$$\sum_{\alpha=1}^{p-1} \frac{1}{a_\alpha^{p+1}} \neq 0. \tag{3.3}$$

Then, the expectation values of the characteristic polynomials lead to (3.1) in the double scaling limit.

We first consider the intersection numbers with one marked point. They are related to the coefficients of $\mathrm{tr}\frac{1}{\Lambda^{\mathrm{m}}}$, in the zero-replica limit $k \to 0$ as we have seen in the previous section. In this limit the matrix Λ can be taken as multiple of the identity $\Lambda = \lambda \cdot \mathbf{1}$. We introduce a coupling constant g as

$$B \to \frac{B}{g} \tag{3.4}$$

$$Z_p = \int dB e^{\frac{1}{(p+1)g^{p+1}}\mathrm{trB}^{p+1} - \frac{\Lambda}{\mathrm{g}}\mathrm{trB}} \tag{3.5}$$

We set $g_s = g^{p+1}$ and tune Λ so that $g_s = \frac{g}{\Lambda}$. Then we obtain, after the shift $B \to 1 + B$,

$$Z_p = \int dB e^{\frac{1}{(p+1)g_s}\mathrm{tr(1+B)}^{p+1} - \frac{1}{\mathrm{g_s}}\mathrm{tr(1+B)}} \tag{3.6}$$

Expanding for small B, we have

$$Z_p = \int dB e^{\frac{p}{2g_s}\mathrm{trB}^2 + \frac{\mathrm{p(p-1)}}{3!\mathrm{g_s}}\mathrm{trB}^3 + \frac{\mathrm{p(p-1)(p-2)}}{4!\mathrm{g_s}}\mathrm{trB}^4 \ldots} \tag{3.7}$$

We now expand in powers of g_s after the replacement $B \to i\sqrt{\frac{g_s}{p}}B$.

Using the replica formula (2.19) for one marked point, we obtain $\lim_{k\to 0} \frac{1}{k} < \mathrm{trB}^4 >= 1, \lim_{\mathrm{k}\to 0} \frac{1}{\mathrm{k}} < (\mathrm{trB}^3)^2 >= 3, \ldots$. Using these values for the products of vertices, we have

$$\lim_{k\to 0} \log Z_p = -\frac{p-1}{24}\left(\frac{g_s}{p}\right) + \frac{(p-3)(p-1)(p+3)(1+2p)(3+2p)}{1920}\frac{g_s^3}{p^3} + O\left(g_s^5\right) \tag{3.8}$$

The coefficients of the above expansion are intersection numbers multiplied by $t_{m,j}$ in (2.15), the first term for genus one, the second for genus two, etc. . . From (2.15), we have $t_{1,0} = -\frac{1}{p}\mathrm{tr}\frac{1}{\Lambda^{p+1}}$. Therefore, the intersection number of one marked point for genus one becomes

$$< \tau_{1,0} >_{g=1}= \frac{p-1}{24}. \tag{3.9}$$

For genus two, we have $t_{3,2} = (-p)^{-2}3(3+p)(3+2p)\mathrm{tr}\frac{1}{\Lambda^{3p+3}}$, and

$$< \tau_{3,2} >_{g=2}= \frac{(p-1)(p-3)(1+2p)}{p\cdot 5!\cdot 4^2\cdot 3}. \tag{3.10}$$

For $g=2$ and $p=2$, we have $t_{4,0} = -\frac{3\cdot 5\cdot 7}{2^3}\mathrm{tr}\frac{1}{\Lambda^9}$ and this gives $< \tau_{4,0} >_{g=2}= \frac{1}{1152}$. The expansion of (3.8) can be obtained for any higher order of genus g by the use of replica formula of (2.19) although the evaluation becomes tedious.

We now turn to the dual model, formulated with $N\times N$ random matrices M; the Fourier transform $\mathrm{U}(s)$ of the one point correlation function is given in (2.23).

We are still in the case in which the external source a_α takes values $p-1$ different values of $a_1, a_2, \ldots, a_{p-1}$ with $\frac{N}{p-1}$ degeneracy.

We have from (2.23),

$$\mathrm{U}(s) = \frac{e^{\frac{s^2}{2}}}{Ns}\oint\frac{du}{2i\pi}e^{us}e^{\sum_{\alpha=1}^{p-1}\frac{N}{p-1}\log(a_\alpha-u-s)-\sum_{\alpha=1}^{p-1}\frac{N}{p-1}\log(a_\alpha-u)} \tag{3.11}$$

Expanding $u(s)$ for small u and s, we have

$$\mathrm{U}(s) = \frac{e^{\frac{s^2}{2}}}{Ns}\oint\frac{du}{2i\pi}e^{-s\sum\frac{N}{(p-1)a_\alpha}+\left(\frac{s^2}{2}+us\right)\left(1-\frac{N}{p-1}\sum\frac{1}{a_\alpha^2}\right)-\sum_{n=3}^{\infty}\frac{N}{n(p-1)a_\alpha^n}((u+s)^n-u^n)} \tag{3.12}$$

With the conditions (3.3), we obtain, after the shift $u\to u-\frac{s}{2}$,

$$\mathrm{U}(s) = \frac{1}{Ns}\int\frac{du}{2i\pi}e^{-\frac{c}{p+1}\left[\left(u+\frac{1}{2}s\right)^{p+1}-\left(u-\frac{1}{2}s\right)^{p+1}\right]} \tag{3.13}$$

where $c = \frac{N}{p-1}\sum\frac{1}{a_\alpha^{p+1}}$.

Expanding the exponent, we obtain

$$\begin{aligned}\mathrm{U}(s) = \frac{1}{Ns}\int\frac{du}{2i\pi}&\exp[-csu^p] \\ &\times\exp\left[-c\left(\frac{p(p-1)}{3!4}s^3u^{p-2}+\frac{p(p-1)(p-2)(p-3)}{5!4^2}s^5u^{p-4}+\cdots\right)\right].\end{aligned} \tag{3.14}$$

This integrals yield Gamma functions after the replacement $u = \left(\frac{t}{cs}\right)^{1/p}$,

$$
\begin{aligned}
\mathrm{U}(s) &= \frac{1}{Nsp\pi} \cdot \frac{1}{(cs)^{1/p}} \int_0^\infty dt t^{\frac{1}{p}-1} e^{-t} \\
&\quad \times e^{-\frac{p(p-1)}{3!4} s^{2+\frac{2}{p}} c^{\frac{2}{p}} t^{1-\frac{2}{p}} - \frac{p(p-1)(p-2)(p-3)}{5!4^2} s^{4+\frac{4}{p}} c^{\frac{4}{p}} t^{1-\frac{4}{p}} + \cdots} \\
&= \frac{1}{Ns\pi} \cdot \frac{1}{(cs)^{1/p}} \left[\Gamma\left(1+\frac{1}{p}\right) - \frac{p-1}{24} y \Gamma\left(1-\frac{1}{p}\right) \right. \\
&\quad + \frac{(p-1)(p-3)(1+2p)}{5! \cdot 4^2 \cdot 3} y^2 \Gamma\left(1-\frac{3}{p}\right) \\
&\quad - \frac{(p-5)(p-1)(1+2p)(8p^2-13p-13)}{7! 4^3 3^2} y^3 \Gamma\left(1-\frac{5}{p}\right) \\
&\quad + (p-7)(p-1)(1+2p)(72p^4 - 298p^3 - 17p^2 + 562p + 281) \\
&\quad\quad \left. \times \frac{1}{9! 4^4 15} y^4 \Gamma\left(1-\frac{7}{p}\right) \cdots \right]
\end{aligned} \tag{3.15}
$$

with $y = c^{\frac{2}{p}} s^{2+\frac{2}{p}}$.

Comparing this expansion with (3.8), we find a common p-dendence, but (3.8) has additional factors. These additional factors should be included in the normalization. The intersection numbers $< \tau_{n,j} >_g$ are the coefficients of (3.15), and then the two results of (3.8) and (3.15) coincide. The expansion of $\mathrm{U}(s)$ is easily obtained up to arbitrary order of genus g since $\mathrm{U}(s)$ is simply given by (3.13). Thus the dual model is simpler than the the partition function of the matrix B.

The expansion of $\mathrm{U}(s)$ in (3.15) is genus expansion. We write this expansion as

$$
\mathrm{U}(s) = \sum_g < \tau_{n,j} >_g \frac{1}{N\pi} \Gamma\left(1 - \frac{1}{p} - \frac{j}{p}\right) c^{\frac{2g-1}{p}} p^{g-1} s^{(2g-1)\left(1+\frac{1}{p}\right)}, \tag{3.16}
$$

where n and j are given by

$$
(p+1)(2g-1) = pn + j + 1. \tag{3.17}
$$

From (3.15), the intersection numbers $< \tau_{n,j} >$ are determined explicitly for arbitrary p.

For $g = 1$ case, by the condition of (3.17), we obtain $n = 1$ and $j = 0$, and

$$
< \tau_{1,0} >_{g=1} = \frac{p-1}{24}. \tag{3.18}
$$

For $g = 2$, we have $n = 3 + \frac{2-j}{p}$. The intersection numbers for arbitrary p become

$$
< \tau_{n,j} >_{g=2} = \frac{(p-1)(p-3)(1+2p)}{p \cdot 5! \cdot 4^2 \cdot 3} \frac{\Gamma\left(1-\frac{3}{p}\right)}{\Gamma\left(1-\frac{1+j}{p}\right)}. \tag{3.19}
$$

For instance, we obtain $< \tau_{3,2} >_{g=2} = \frac{(p-1)(p-3)(1+2p)}{p5! \cdot 4^2 \cdot 3}$, which agrees with the result of (3.10) evaluated from (3.8). For $p = 2$, we have from this formula, $< \tau_{4,0} >_{g=2} = \frac{1}{(24)^2 2!}$, since the ratio of the gamma functions becomes -2.

For $g = 3$, we have the intersection numbers for arbitrary p with $n = 5 + \frac{4-j}{p}$,

$$< \tau_{n,j} >_{g=3} = \frac{(p-5)(p-1)(1+2p)(8p^2-13p-13)}{p^2 \cdot 7! \cdot 4^3 \cdot 3^2} \frac{\Gamma\left(1-\frac{5}{p}\right)}{\Gamma\left(1-\frac{1+j}{p}\right)}. \tag{3.20}$$

For $g = 4$, we have the intersection numbers with thecondition $n = 7 + \frac{6-j}{p}$,

$$< \tau_{n,j} >_{g=4} = \frac{(p-7)(p-1)(1+2p)(72p^4-298p^3-17p^2+562p+281)}{p^3 \cdot 9! \cdot 4^4 \cdot 15} \frac{\Gamma\left(1-\frac{7}{p}\right)}{\Gamma\left(1-\frac{1+j}{p}\right)}. \tag{3.21}$$

Thus the expansion of U(s) gives the intersection numbers of one marked point for arbitrary p in the case of fixed g. We have given the explicit expressions up to order $g = 4$ only. Also the integral representation of U(s) in (3.13) gives the intersection numbers for arbitrary g for fixed p. As shown before in [7], the intersection numbers for $p = 2$ becomes

$$< \tau_{n,0} >_g = \frac{1}{(24)^g g!}, \qquad (n = 3g-2). \tag{3.22}$$

For $p = 3$,

$$< \tau_{n,j} >_g = \frac{1}{(12)^g g!} \frac{\Gamma\left(\frac{1+g}{3}\right)}{\Gamma\left(\frac{2-j}{3}\right)}, \tag{3.23}$$

with $3n = 8g - 5 - j$.

We now consider the interesting limit $p \to -1$. We consider the spin index j as $j = 0$ in this limit. From the condition of (3.17), we obtain $n = 1$. The intersection number of $p = -1$ case is then $< \tau_{1,0} >_g$, which we write simply as $< \tau >_g$ in the following. From the previous evaluations in (3.18)–(3.21), we obtain in the limit $p \to -1$,

$$\begin{aligned} &< \tau >_{g=1} = \frac{p-1}{24} \to -\frac{1}{12} \\ &< \tau >_{g=2} = \frac{(p-1)(p-3)(1+2p)}{5!4^2} \frac{\Gamma\left(1-\frac{3}{p}\right)}{\Gamma\left(1-\frac{1}{p}\right)} \to -\frac{1}{120} \\ &< \tau >_{g=3} \to -\frac{1}{252}, \qquad\qquad < \tau >_{g=4} \to -\frac{1}{240}. \end{aligned} \tag{3.24}$$

These numbers are the Euler characteristics $\chi(M_{g,1})$ [3, 11].

$$\chi(M_{g,1}) = \zeta(1-2g) = -\frac{B_{2g}}{2g} \tag{3.25}$$

where ζ is the Riemann zeta-function and B_{2g} is the Bernoulli number; $B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, B_8 = \frac{1}{30} \cdots$.

The logarithmic term of the Penner model follows indeed in the limit $p \to -1$ of (3.13). Then computing U(s) for $p \to -1$, one obtains the Euler characteristics.

JHEP04(2009)110

In the limit $p \to -1$, $c = \frac{N}{p-1} \sum \frac{1}{a_\alpha{}^{p+1}}$ is N, and from (3.13), $\mathrm{U}(s)$ is given by

$$\begin{aligned} \mathrm{U}(s) &= \frac{1}{Ns} \int \frac{du}{2i\pi} e^{-N \log \frac{u+\frac{1}{2}s}{u-\frac{1}{2}s}} \\ &= \frac{1}{N} \int \frac{du}{2i\pi} \left(\frac{u-\frac{1}{2}}{u+\frac{1}{2}} \right)^N \end{aligned} \tag{3.26}$$

Setting

$$\frac{u-1}{u+1} = e^{-y}, \qquad \left(u = \frac{1+e^{-y}}{1-e^{-y}} \right) \tag{3.27}$$

one has

$$\mathrm{U}(s) = -\frac{1}{N} \int \frac{dy}{2\pi} \frac{e^{-y}}{(1-e^{-y})^2} e^{-Ny} = \int_0^\infty \frac{dy}{2\pi} \frac{e^{-Ny}}{1-e^{-y}} \tag{3.28}$$

Noting that

$$\frac{1}{1-e^{-t}} = \sum_{n=0}^{\infty} B_n \frac{t^{n-1}}{n!} \tag{3.29}$$

we obtain $\mathrm{U}(s)$ as

$$\begin{aligned} \mathrm{U}(s) &= \frac{1}{N} \int_0^\infty dt \frac{1}{1-e^{-\frac{t}{N}}} e^{-t} = \sum_{n=0}^{\infty} \frac{B_n}{n} \left(\frac{1}{N} \right)^n \\ &= 1 - \frac{1}{2N} + \frac{1}{12N^2} - \frac{1}{120} \frac{1}{N^4} + \frac{1}{252} \frac{1}{N^6} + \cdots \end{aligned} \tag{3.30}$$

Then we obtain the genus g, orbifold Euler characteristics $\chi(M_{g,1}) = \zeta(1-2g) = -\frac{1}{2g} B_{2g}$ from the term of order $1/N^{2g}$. Thus the analytic continuation for negative p holds for the dual model.

4 The n-point correlation functions

We consider the two-point correlation function $\mathrm{U}(s_1, s_2)$ defined in (2.1). Noting that the two terms of the determinant in (2.23) become, after the shift $u_i \to u_i - \frac{s_i}{2}$, $s_i \to \frac{s_i}{N}$,

$$\begin{aligned} &\frac{1}{u_1 - u_2 + \frac{1}{2N}(s_1+s_2)} \frac{1}{u_1 - u_2 - \frac{1}{2N}(s_1+s_2)} \\ &\quad = \left(\frac{1}{u_1 - u_2 - \frac{1}{2N}(s_1+s_2)} - \frac{1}{u_1 - u_2 + \frac{1}{2N}(s_1+s_2)} \right) \frac{N}{s_1+s_2} \end{aligned} \tag{4.1}$$

we write it as

$$\frac{1}{u_1 - u_2 + \frac{1}{2N}(s_1+s_2)} \frac{1}{u_1 - u_2 - \frac{1}{2N}(s_1+s_2)} = \frac{N}{s_1+s_2} \int_0^\infty dx e^{-x(u_1-u_2)} \mathrm{sh}\left(\frac{x}{2N}(s_1+s_2) \right) \tag{4.2}$$

We have at the same p-th critical point defined for the one point function,

$$\mathrm{U}(s_1,s_2) = \frac{2N}{s_1+s_2}\frac{1}{(2\pi i)^2}\int_0^\infty dx\int du_1du_2 \mathrm{sh}\left(\frac{1}{2N}x(s_1+s_2)\right) \times\exp\left[-\frac{N}{p^2-1}\sum\frac{1}{a_\alpha^{p+1}}\left(\sum_i\left(u_i+\frac{1}{2N}s_i\right)^{p+1}-\sum_i\left(u_i-\frac{1}{2N}s_i\right)^{p+1}\right)\right] \tag{4.3}$$

We use the notation, $c=\sum_\alpha\frac{1}{a_\alpha^{p+1}}$. After the change of variables $u_i\to iv_i$ (i=1,2), the rescalings $v_i\to(\frac{p-1}{pcs_i})^{\frac{1}{p}}v_i$, and $x\to(\frac{pcs_1}{p-1})^{\frac{1}{p}}x$, we obtain

$$\mathrm{U}(s_1,s_2) = \frac{2N'}{s_1+s_2}\left(\frac{1}{s_2}\right)^{\frac{1}{p}}\int_0^\infty dx\int_{-\infty}^\infty\frac{dv_1dv_2}{(2\pi)^2}\mathrm{sh}\left(\frac{x}{2N'}s_1^{\frac{1}{p}}(s_1+s_2)\right) e^{-ixv_1+ixv_2(\frac{s_1}{s_2})^{\frac{1}{p}}}\prod_{i=1}^2 G(v_i) \tag{4.4}$$

where we used $N'=N\left(\frac{p-1}{pc}\right)^{\frac{1}{p}}$, and $\left[\frac{p}{2}\right]=\frac{p}{2}$ for even p and $\left[\frac{p}{2}\right]=\frac{p-1}{2}$ for odd p. The factor $G(v_i)$ is given by

$$G(v_i)=\exp\left[-\frac{i^p}{p}v_i^p-i^p\sum_{m=1}^{[\frac{p}{2}]}\frac{(-1)^m(p-1)!}{(2m+1)!2^{2m}(p-2m)!N'^{2m}}s_i^{\left(2+\frac{2}{p}\right)m}v_i^{p-2m}\right]. \tag{4.5}$$

The genus g of the terms in the expansion (4.5) is given by the exponent of $\frac{1}{N'^{2g}}$. We are interested in the terms of type $s_1^{n_1+\frac{m_1}{p}}s_2^{n_2+\frac{m_2}{p}}$ in (4.4). The correspondence with the variable $t_{n,m}\sim\mathrm{tr}\frac{1}{\Lambda^{pn+m+1}}$ is

$$s^{n+\frac{m+1}{p}}\sim t_{n,m} \tag{4.6}$$

Thus we obtain the intersection numbers $<\tau_{n_1,m_1}\tau_{n_2,m_2}>_g$ from the coefficients of the terms $s_1^{n_1+\frac{m_1+1}{p}}s_2^{n_2+\frac{m_2+1}{p}}$. For instance, we have from $s_1^{\frac{1}{p}}s_2^{2+\frac{1}{p}}$ in (4.4),

$$<\tau_{0,0}\tau_{2,0}>_{g=1}=\frac{p-1}{24} \tag{4.7}$$

This value coincides with $<\tau_{1,0}>_{g=1}=\frac{p-1}{24}$, and we have

$$<\tau_{0,0}\tau_{2,0}>_{g=1}=<\tau_{1,0}>_{g=1} \tag{4.8}$$

which is consistent with the string equation. Indeed the generating function F for the intersection numbers satisfies the string equation [4],

$$\frac{\partial F}{\partial t_{0,0}}=\frac{1}{2}\sum_{m,m'=0}^{p-2}\eta^{mm'}t_{0,m}t_{0,m'}+\sum_{n=1}^\infty\sum_{m=0}^{p-2}t_{n+1,m}\frac{\partial F}{\partial t_{n,m}} \tag{4.9}$$

where the metric $\eta^{mm'}=\delta_{m+m',p-2}$.

If we substitute $F = at_{1,0}$ (a is some constant) in the second term, then we have $\frac{\partial F}{\partial t_{0,0}} = t_{2,0}\frac{\partial F}{\partial t_{1,0}} = at_{2,0}$, which yields $at_{0,0}t_{2,0}$ after integration. Thus we have $<\tau_{0,0}\tau_{2,0}>=<\tau_{1,0}>$ from the string equation (4.9).

Note that in the two point correlation (two marked points), there is no genus zero contribution to the intersection numbers, an easy consequence of (4.4).

It is useful to define the higher Airy functions $\phi_p(x)$ by

$$\phi_p(x) = \int \frac{dv}{2\pi} e^{-\frac{i^p}{p}v^p + ixv} \tag{4.10}$$

When $p = 3$, it reduces to the usual Airy function $A_i(x)$,

$$\phi_3(x) = A_i(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dv e^{\frac{i}{3}v^3 + ixv} \tag{4.11}$$

which satisfies the differential equation,

$$\phi_3(x)'' = x\phi_3(x) \tag{4.12}$$

For general p, we have

$$\frac{d^{p-1}\phi_p(x)}{dx^{p-1}} = x\phi_p(x) \tag{4.13}$$

Expanding in powers of $\frac{1}{N'}$, after integration by parts for the function $\phi_p(x)$, we obtain all intersection numbers from (4.4) for two marked points. In the case $p = 2$, the function $\phi_p(x)$ is Gaussian, and we are led to

$$\begin{aligned} \mathrm{U}(s_1, s_2) &= \frac{2N}{(s_1+s_2)\sqrt{s_2}} e^{\frac{1}{24N^2}(s_1^3+s_2^3)} \int_0^\infty dx \,\mathrm{sh}\left(x\frac{\sqrt{s_1}}{2N}(s_1+s_2)\right) e^{-\frac{1}{2}\frac{s_1+s_2}{s_2}x^2} \\ &= \frac{N}{s_1+s_2} e^{\frac{1}{24N^2}(s_1+s_2)^3} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(2m+1)} \left(\frac{s_1 s_2(s_1+s_2)}{8N^2}\right)^m \sqrt{s_1 s_2} \end{aligned} \tag{4.14}$$

which has been obtained in [5, 18]. The intersection numbers are given by

$$\mathrm{U}(s_1, s_2) = \sum_{n_1, n_2} <\tau_{n_1,0}\tau_{n_2,0}>_g \frac{s_1^{n_1} s_2^{n_2}}{N^{2g}} \tag{4.15}$$

where the genus g is specified by

$$n_1 + n_2 = 3g - 2. \tag{4.16}$$

For the $p = 3$ case, we have

$$\mathrm{U}(s_1, s_2) = \frac{2N'}{s_1+s_2}\left(\frac{1}{\sqrt{s_2}}\right)^{\frac{1}{3}} \int_0^\infty dx \,\mathrm{sh}\left(\frac{x}{2N'} s_1^{\frac{1}{3}}(s_1+s_2)\right) A_i(x) A_i\left(-x\left(\frac{s_1}{s_2}\right)^{\frac{1}{3}}\right) \tag{4.17}$$

which gives the intersection numbers of two-marked points. For instance, we obtain

$$<\tau_{0,0}\tau_{2,0}>_{g=1} = \frac{1}{12}, \quad <\tau_{1,0}^2>_{g=1} = \frac{1}{12} \tag{4.18}$$

In general, the integral formula of (2.23) gives the n-point correlation function $\mathrm{U}(s_1,\ldots,s_n)$. In appendix A, we compute the intersection numbers for three marked points from the three point function$\mathrm{U}(s_1,s_2,s_3)$. In appendix B, the intersection numbers with four marked points are computed from the four point correlation function.

The intersection numbers for the primary field U_j of (2.16), in the genus zero case, are particularly important, since they have algebraic structures related to a super-conformal field theory [4, 13]. They are expressed as $< \prod_{m=1}^{n} \tau_{0,q_m} >$.

In appendix A, we compute the intersection numbers (A.8) as

$$< \tau_{0,q_1}\tau_{0,q_2}\tau_{0,q_3} >_{g=0} = \delta_{q_1+q_2+q_3,p-2} \tag{4.19}$$

From this result, the free energy F follows

$$F = \sum < \tau_{0,q_1}\tau_{0,q_2}\tau_{0,q_3} >_{g=0} t_{0,q_1}t_{0,q_2}t_{0,q_3} + O(t^4) \tag{4.20}$$

To make the algebraic structure more explicit, we define the structure constants C_{ijk} by

$$C_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \tag{4.21}$$

where

$$t_i = t_{0,i-1}, \qquad (i = 1,\ldots,p-1) \tag{4.22}$$

For instance for p=5, we have

$$F = \frac{1}{2}t_{0,0}^2 t_{0,3} + t_{0,0}t_{0,1}t_{0,2} + \frac{1}{3!}t_{0,1}^3 + O(t^4) \tag{4.23}$$

and the structure constants are $C_{114} = C_{123} = C_{222} = 1$. From these structure constants, following [4] one can construct the super potential W (see appendix C). The free energy F, which generates the intersection numbers of the moduli space of p-spin curves, has been conjectured by Witten to be solution of the Gelfand-Dikii hierarchy [4]. We present in appendix C, this Gelfand-Dikii hierarchic equations as well as the construction of the super potential for the primary fields.

Thus, we find that the intersection numbers, derived from the integral representation of $\mathrm{U}(s_1,\ldots,s_n)$, satisfy indeed the Gelfand-Dikii equations.

Returning to Witten's conjecture, we note that the definition of the intersection numbers by the vector bundle integration over the compactified moduli space $\bar{M}_{g,n}$ in (2.16), is similar in structure to the integral representation for $\mathrm{U}(s_1,\ldots,s_n)$ of (2.23), although $\mathrm{U}(s_1,\ldots,s_n)$ involves a summation over all genuses g, for fixed n-marked points.

5 Time-dependent Gaussian matrix model

Let us briefly recall how one shows that the time-dependent one matrix model is equivalent to a time-independent two-matrix model when the distributions are Gaussian. (In an older work we had considered the time-dependent Gaussian matrix problem, and computed time-dependent correlation functions [20]).

The time-dependent Gaussian matrix model is the partition function (i.e. the Euclidean path integral) for the matrix quantum mechanics, with action

$$S = \int dt \frac{1}{2} \mathrm{tr}(\dot{M}^2 + M^2) \tag{5.1}$$

where M is an $N \times N$ Hermitian matrix; (the dot stands for time derivative). This model, at criticality, is known to describe gravity coupled to matter of central charge $c = 1$ [21].

The time-dependent correlation function is defined by

$$\rho(\lambda, \mu; t) = < \frac{1}{N} \mathrm{tr}\delta(\lambda - M(t_1)) \frac{1}{N} \mathrm{tr}\delta(\mu - M(t_2)) > \tag{5.2}$$

and, from time-translation invariance, is function of $t = |t_2 - t_1|$. The Fourier transform of this quantity is

$$\mathrm{U}(\alpha, \beta) = \frac{1}{N^2} < \mathrm{tr} e^{i\alpha M(t_1)} \mathrm{tr} e^{i\beta M(t_2)} > \tag{5.3}$$

This correlation function may easily be reduced to the correlation function of the time-independent two matrix model in the Gaussian ensemble [20]. Indeed this harmonic oscillator quantum mechanics leads to

$$\mathrm{U}(\alpha, \beta) = \frac{1}{N^2} \left(\frac{e^t}{\mathrm{sinh} t} \right)^{N^2/2} \int dAdB \mathrm{tr} e^{\mathrm{i}\alpha\mathrm{A}} \mathrm{tr} e^{\mathrm{i}\beta\mathrm{B}} \mathrm{e}^{-\frac{1}{2}\mathrm{shttr}[(\mathrm{A}^2+\mathrm{B}^2)\mathrm{e}^{\mathrm{t}} - 2\mathrm{AB}]} \tag{5.4}$$

Rescaling of A,B,α and β by a factor $\sqrt{e^{-t}\mathrm{sinh}t}$, we obtain a time-independent two-matrix model

$$\mathrm{U}(\alpha, \beta) = \frac{1}{Z} \int dAdB \mathrm{tr} e^{i\alpha A} \mathrm{tr} e^{\mathrm{i}\beta\mathrm{B}} \mathrm{e}^{-\frac{1}{2}\mathrm{tr}(\mathrm{A}^2+\mathrm{B}^2-2\mathrm{cAB})}, \tag{5.5}$$

with a coupling constant

$$c = e^{-t}.$$

For convenience, we denote A and B by M_1 and M_2 in the following.

6 Duality formula for the two-matrix model

We consider the correlation function of the characteristic polynomials in the two-matrix model,

$$J = < \prod_{\alpha=1}^{k_1} \det(\lambda_\alpha - M_1) \prod_{\beta=1}^{k_2} \det(\mu_\beta - M_2) > \tag{6.1}$$

where the average is perfomed over a two-matrix Gaussian distribution with an external source A acting on one of the two $N \times N$ matrices,

$$P(M_1, M_2) = \frac{1}{Z} e^{-\frac{1}{2}\mathrm{tr}M_1^2 - \frac{1}{2}\mathrm{tr}M_2^2 - c\mathrm{tr}M_1M_2 - \mathrm{tr}M_1 A} \tag{6.2}$$

The external source A will be used here again mean to tune a p-spin structure in the moduli space. Note that when $t \to \infty$ the parameter $c = e^{-t}$ vanishes and the two matrices M_1 and M_2 decouple.

From (6.1) we shall determine a new dual model of Kontsevich type in the large N limit. The duality formula for J is obtained by the use of Grassmann variables as in the one matrix case [6, 16], but for the two matrix case a new structure appears.

Let us introduce the Grassmann variables ψ_i^α and χ_i^β, where $\alpha = 1, \ldots, k_1$, and $\beta = 1, \ldots, k_2$. Then

$$J =< \int d\bar{\psi} d\psi d\bar{\chi} d\chi e^{N[\bar{\psi}_\alpha(\lambda_\alpha - M_1)\psi_\alpha + \bar{\chi}_\beta(\mu_\beta - M_2)\chi_\beta]} > \tag{6.3}$$

Since the probability P is Gaussian, one can integrate out the matrices M_1 and M_2. This generates four-fermion terms that may be disentangled with the help of three auxiliary matrices: B_1 a $k_1 \times k_1$ Hermitian matrix, B_2 a $k_2 \times k_2$ Hermitian matrix and D a complex $k_1 \times k_2$ rectangular matrix. The identities

$$e^{-\frac{N}{2(1-c^2)}\bar{\psi}\psi\bar{\psi}\psi} = \int dB_1 e^{-\frac{N}{2}\mathrm{tr}B_1^2 + \frac{iN}{\sqrt{1-c^2}}\mathrm{tr}B_1\bar{\psi}\psi} \tag{6.4}$$

$$e^{-\frac{N}{2(1-c^2)}\bar{\chi}\chi\bar{\chi}\chi} = \int dB_2 e^{-\frac{N}{2}\mathrm{tr}B_2^2 + \frac{iN}{\sqrt{1-c^2}}\mathrm{tr}B_2\bar{\chi}\chi} \tag{6.5}$$

$$e^{\frac{Nc}{1-c^2}\bar{\psi}\chi\bar{\chi}\psi} = \int dD dD^\dagger e^{-N\mathrm{tr}D^\dagger D + N\sqrt{\frac{c}{1-c^2}}\mathrm{tr}(D\bar{\psi}\chi + D^\dagger\bar{\chi}\psi)} \tag{6.6}$$

allow to represent J as

$$J = \int dB_1 dB_2 dD^\dagger dD e^{-\frac{N}{2}\mathrm{tr}(B_1^2 + B_2^2 + 2D^\dagger D)}$$
$$\times \prod_{i=1}^N \det \begin{pmatrix} \left(\lambda_\alpha - \frac{a_i}{1-c^2}\right)\delta_{\alpha,\alpha'} + \frac{i}{\sqrt{1-c^2}}B_1 & \sqrt{\frac{c}{1-c^2}}D \\ \sqrt{\frac{c}{1-c^2}}D^\dagger & \left(\mu_\beta + \frac{c}{1-c^2}a_i\right)\delta_{\beta,\beta'} + \frac{i}{\sqrt{1-c^2}}B_2 \end{pmatrix} \tag{6.7}$$

After the shift $B_1 \to B_1 + i\sqrt{1-c^2}\lambda_{\alpha,\alpha'}\delta_{\alpha,\alpha'}$ and $B_2 \to B_2 + i\sqrt{1-c^2}\mu_{\beta,\beta'}\delta_{\beta,\beta'}$, we obtain the dual expression for J:

$$J = C\int dB_1 dB_2 dD^\dagger dD e^{-\frac{N}{2}\mathrm{tr}(B_1^2+B_2^2+2D^\dagger D) - iN\sqrt{1-c^2}\mathrm{tr}B_1\Lambda_1 - iN\sqrt{1-c^2}\mathrm{tr}B_2\Lambda_2}$$
$$\times e^{-\sum_{i=1}^N \mathrm{trlog}(1-X_i)}, \tag{6.8}$$

where the matrices X_i are defined by

$$X_i = \begin{pmatrix} i\sqrt{1-c^2}\frac{B_1}{a_i} & \sqrt{c(1-c^2)}\frac{D}{a_i} \\ -\frac{\sqrt{c(1-c^2)}}{c}\frac{D^\dagger}{a_i} & -\frac{i\sqrt{1-c^2}}{c}\frac{B_2}{a_i} \end{pmatrix}. \tag{6.9}$$

We now expand $\log(1-X_i)$ in powers of X_i. Let us first consider the case of a source matrix A mutiple of the identity: $a_i = a,\ i = 1\cdots N$. Imposing the constraint

$$a = (1-c^2) \tag{6.10}$$

the quadratic term $\mathrm{tr}B_1^2$ cancels with the one coming from the expansion of $\log(1-X)$. This critical constraint corresponds to the edge of the the spectrum for the matrix M_1. Note that the B_2^2 term is not cancelled at this critical point because of the coupling c.

Given the factor N in the exponent, the edge scaling limit under consideration corresponds to

$$B_1 \sim O\left(N^{-\frac{1}{3}}\right), B_2 \sim O\left(N^{-\frac{1}{2}}\right), D \sim O\left(N^{-\frac{1}{3}}\right) \tag{6.11}$$

in the large N limit. In this limit most terms disappear; for instance

$$N\mathrm{tr}\left(D^\dagger D B_2\right) \sim N^{-\frac{1}{6}} \tag{6.12}$$

is negligible. Then, in the large N limit (6.11), we obtain the partition function Z, i.e. J after dropping the negligible terms,

$$Z = \int dB_1 dB_2 dD^\dagger dD e^{-iN\mathrm{tr}B_1\Lambda_1 - iN\mathrm{tr}B_2\Lambda_2 + \frac{i}{3}N\mathrm{tr}B_1^3 - \frac{N}{2}\left(1-\frac{1}{c^2}\right)\mathrm{tr}B_2^2 + iN\mathrm{tr}(DD^\dagger B_1)} \tag{6.13}$$

Since the matrix B_2 matrix is decoupled we can integrate it out. Then, dropping the decoupled part, we find the partition function

$$Z = \int dB_1 dD^\dagger dD e^{-i\mathrm{tr}B_1\Lambda_1 + \frac{i}{3}\mathrm{tr}B_1^3 + i\mathrm{tr}DD^\dagger B_1} \tag{6.14}$$

where we have absorbed the powers of N given by the scaling (6.11). We may now integrate out the matrices D and $D^\dagger$; this yields a one matrix integral with a logarithmic potential,

$$Z = \int dB_1 e^{\frac{i}{3}\mathrm{tr}B_1^3 - k_2\mathrm{tr}\log B_1 - i\mathrm{tr}B_1\Lambda_1}. \tag{6.15}$$

The appearance of a logarithmic term is a characteristic of models with central charge equal to one.

We now consider the free energy of this logarithmic Kontsevich model (p=2) (6.15). Three different, but consistent, methods will be used. (For convenience k_2 is denoted as q in what follows.)

i) HarishChandra-Itzykson-Zuber method After use of the HarishChandra-Itzykson-Zuber formula, the partition function Z is given by

$$\begin{aligned} Z &= \int dB e^{\frac{i}{3}\mathrm{tr}B^3 - q\mathrm{tr}\log B - i\mathrm{tr}B\Lambda^2} \\ &= \frac{1}{\Delta(l^2)} \int \prod_{i=1}^{k_1} dx_i \Delta(x) \prod_{i=1}^{k_1} e^{-ix_i l_i^2 + \frac{i}{3}x_i^3 - q\log x_i} \end{aligned} \tag{6.16}$$

where the x_i's are the eigenvalues of B, the l_i the eigenvalues of Λ, and $\Delta(x)$ the Vandermonde determinant $\Delta(x) = \prod_{i<j}(x_i - x_j)$. It may be replaced in the integrand by the Vandermonde of differential operators $\frac{\partial}{\partial l_i^2}$,

$$Z = \prod \frac{1}{(l_i^2 - l_j^2)} \left(\frac{\partial}{\partial l_i^2} - \frac{\partial}{\partial l_j^2}\right) \prod \zeta(l_i) \tag{6.17}$$

where

$$\zeta(l_i) = \int dx e^{\frac{i}{3}x^3 - ixl_i^2 - q\log x}. \tag{6.18}$$

Rescaling $x_i \to x_i/2^{1/3}$ and $l_i \to l_i/2^{1/3}$, and with the change $l \to il$, we have

$$\zeta(l) = \frac{e^{-\frac{1}{3}l^3}}{l^{q+\frac{1}{2}}} \int dx e^{-\frac{1}{2}x^2 + \frac{i}{6l^{3/2}}x^3 - q\log\left(1+\frac{x}{il^{3/2}}\right)} \tag{6.19}$$

Expanding for large l, we obtain

$$\begin{aligned}\log Z &= -\left[\frac{1}{6}t_1^3 + \frac{1}{24}t_3 + qt_2t_1 + \frac{1}{2}q^2t_3\right] + \left[\frac{1}{6}t_1^3t_3 + \frac{1}{48}t_3^2 + \frac{1}{8}t_1t_5 + \frac{2}{3}qt_6 + qt_1^2t_4\right.\\ &\left. + qt_1t_2t_3 + \frac{1}{6}qt_2^3 + \frac{3}{2}q^2t_1t_5 + \frac{1}{4}q^2t_3^2 + q^2t_2t_4 + \frac{2}{3}q^3t_6\right] + O\left(\frac{1}{l^9}\right)\end{aligned} \tag{6.20}$$

where we have used the moduli parameters

$$t_n = \sum_{i=1}^{k_1} \frac{1}{l_i^n} \tag{6.21}$$

When $q \to 0$, we recover the result of the one-matrix Kontsevich model.

For the relation to the genus g, we have to identify the powers of $\frac{1}{N^2}$. In the limit in which

$$k_1 \sim q \sim N, l_i \sim N^{\frac{1}{3}} \tag{6.22}$$

we find

$$t_1 \sim O\left(N^{\frac{2}{3}}\right), t_n \sim O\left(N^{1-\frac{1}{3}n}\right) \tag{6.23}$$

The genus expansion of the free energy

$$\log Z = \sum_{g=0}^{\infty} a_g N^{2-2g} \tag{6.24}$$

follows from this limit. For instance, t_1^3,qt_1t_2, and q^2t_3 are contributions to genus zero, and t_3 to genus one.

ii) replica method We return to the integral (6.16). After the shift $B \to B + \Lambda$, which eliminates the terms linear in B, one can expand for large Λ. Then the logarithmic potential, $\mathrm{tr}\log(\mathrm{B}+\Lambda)$ expanded in powers of Λ^{-1} yields $\mathrm{trB^n}$ vertices. The situation is similar to that of the generalized Kontsevich model where the $\mathrm{trB^{p+1}}$ terms led to the moduli space of p-th curves, and spin structures appeared. The occurence of t_2,t_4 and t_5 indicates this fact.

We consider the moduli space for Riemann surfaces with marked points. Although there is a logarithmic potential, the model allows one to consider marked points, whose number is equal to the number of t_n.

We have developed the replica method $k_1 \to 0$ in a previous article [6]. Any average of the products of vertices $\mathrm{trB^n}$ are obtained in the replica limit $k_1 \to 0$, where B is a $k_1 \times k_1$ Hermitian matrix; for a Gaussian ensemble, the average is given by the replica limit formula (2.19).

JHEP04(2009)110

Expanding the logarithmic term, after the shift $B \to B + \Lambda$, and the use of the formula (2.19), we obtain the replica limit, which gives the required intersection numbers with one marked point. Indeed it leads easily to

$$\log Z = -\left(\frac{1}{24} + \frac{1}{2}q^2\right) t_3 + \left(\frac{2}{3}q + \frac{2}{3}q^3\right) t_6 + O\left(\frac{1}{\Lambda^9}\right) \tag{6.25}$$

This result agrees completely with the expression (6.20) for one marked point.

iii) differential equation of Virasoro type Since the free energy $\log Z$ is expressed in terms of the moduli parameters t_n, as was the case in the original Kontsevich model (q=0 case), it is natural to investigate here again the KdV-like differential equations or string equations. Although there is a logarithmic potential one may use a Schwinger-Dyson equation [22, 23].

We first consider the simple case, $k_1 = 1$, a one by one matrix, i.e. a c-number. Denoting $l_1 = x$, one finds

$$Z = e^{-\frac{1}{3}x^3} \frac{1}{x^{q+\frac{1}{2}}} g(x). \tag{6.26}$$

The Schwinger-Dyson (Virasoro) equation follows from the identity

$$\int dB \frac{\partial}{\partial B} e^{\frac{i}{3}\mathrm{trB}^3 - i\mathrm{trB}\Lambda^2 - q\mathrm{tr}\log B} = 0 \tag{6.27}$$

The matrix B is the replaced by $\frac{\partial}{\partial \Lambda^2}$. For the logarithmic potential, it means $(\frac{\partial}{\partial \Lambda^2})^{-1}$. Therefore we need to apply a differential operator in order to get rid of this integral. We find easily that when B is just a real number ($k_1 = 1$), the function $g(x)$ satisfies a third order differential equation,

$$\left[\left(\frac{\partial}{x\partial x}\right)^3 - 2 + 2q - x\frac{\partial}{\partial x}\right] \frac{e^{-\frac{1}{3}x^3}}{x^{q+\frac{1}{2}}} g(x) = 0 \tag{6.28}$$

i.e.

$$\begin{aligned}
&(-(1+2q)(5+2q)(9+2q) + (-10 - 48q - 24q^2)x^3)g \\
&\quad +((66 + 96q + 24q^2)x + (24 + 48q)x^4 + 16x^7)g'(x) \\
&\qquad +(-36x^2 - 24qx^2 - 24x^5)g'' + 8x^3 g'''(x) = 0.
\end{aligned} \tag{6.29}$$

This provides the large x expansion,

$$\begin{aligned}
g(x) = &-1 + \frac{1}{x^3}\left(\frac{5}{24} + q + \frac{q^2}{2}\right) - \frac{1}{x^6}\left(\frac{385}{1152} + \frac{73}{24}q + \frac{161}{48}q^2 + \frac{7}{6}q^3 + \frac{1}{8}q^4\right) \\
&+\frac{1}{x^9}\left(\frac{85085}{82944} + \frac{6259}{384}q + \frac{58057}{2304}q^2 + \frac{2075}{144}q^3 + \frac{725}{192}q^4 + \frac{11}{24}q^5 + \frac{1}{48}q^6\right) + O\left(\frac{1}{x^{12}}\right).
\end{aligned} \tag{6.30}$$

For $k_1 = 2$, a two by two matrix, we denote $x = l_1$ and $y = l_2$; then

$$\left[\left(\frac{\partial}{x\partial x}\right)^3 + \frac{\partial}{x\partial x}\left(\frac{2}{x^2 - y^2}\left(\frac{\partial}{x\partial x} - \frac{\partial}{y\partial y}\right)\right) - 2 + 2q - x\frac{\partial}{\partial x}\right] \frac{e^{-\frac{1}{3}(x^3+y^3)}}{(xy)^{q+\frac{1}{2}}(x+y)} g(x,y) = 0 \tag{6.31}$$

The solution, after symmetrization over x and y, agrees with the expression (6.20).

When x is small, the equation (6.29) leads to a different series expansion. The differential equation for g in (6.29) has three different solutions,

$$g(x) \sim x^{q+\frac{1}{2}}, g(x) \sim x^{q+\frac{5}{2}}, g(x) \sim x^{q+\frac{9}{2}} \tag{6.32}$$

For small x, we have from the first solution, noting that $l = x$,

$$Z = e^{-\frac{1}{3}l^3}\left(1 + \frac{1}{3}l^3 + \left(-\frac{1}{24}q + \frac{7}{72}\right)l^6 + O\left(l^9\right)\right) \tag{6.33}$$

This solution for small l is in a different phase from Kontsevich's phase; it may be related to one of the two phases of the unitary matrix model [23–25].

We now consider the case $p > 2$. After the integration over the D-fields within the matrix X, we also obtain the logarithmic term $\mathrm{tr}\log B$, but corrections appear. By tuning the external source with the conditions (3.3), we obtain

$$Z = \int dX e^{-\frac{1}{p+1}\mathrm{tr}X^{p+1} + \mathrm{tr}X\Lambda^p} \tag{6.34}$$

where X is given by

$$X = \begin{pmatrix} B & D \\ D^\dagger & 0 \end{pmatrix}. \tag{6.35}$$

where we have scaled out the factors $\sqrt{1-c^2}$ in X_i of (6.8), and put $B = B_1$ and $B_2 = 0$. We expand the potential,

$$\begin{aligned} \mathrm{tr}X^{p+1} &= \mathrm{tr}B^{p+1} + (p+1)\mathrm{tr}DD^\dagger B^{p-1} + \frac{1}{2}(p+1)(p-2)\mathrm{tr}\left(DD^\dagger\right)^2 B^{p-3} \\ &+ \frac{1}{6}(p+1)(p-3)(p-4)\mathrm{tr}\left(DD^\dagger\right)^3 B^{p-5} + \cdots . \end{aligned} \tag{6.36}$$

For $p = 3$, we obtain,

$$Z = \int dBdD^\dagger dD e^{-\left[\frac{1}{4}\mathrm{tr}B^4 + \mathrm{tr}DD^\dagger B^2 + \frac{1}{2}\mathrm{tr}\left(DD^\dagger\right)^2\right] + \mathrm{tr}B\Lambda^3}. \tag{6.37}$$

The integration of the D-field can not be done explicitly for the general values of k_1 and k_2 (B is a $k_1 \times k_1$ Hermitian matrix and D is a $k_1 \times k_2$ complex matrix). We make here a perturbation for the large B in lower orders. Expanding the term $\exp(-\frac{1}{2}\mathrm{tr}(DD^\dagger)^2)$, we find

$$Z = \int dB e^{+\mathrm{tr}B\Lambda^3 - \frac{1}{4}\mathrm{tr}B^4 - 2k_2\mathrm{tr}\log B - \frac{1}{2}k_2^2\left(\mathrm{tr}\frac{1}{B^2}\right)^2 - \frac{1}{2}k_2\mathrm{tr}\frac{1}{B^4} + O\left(\frac{1}{B^8}\right)}. \tag{6.38}$$

For $p = 4$ case, the partition function Z becomes similarly

$$\begin{aligned} Z &= \int dBdDdD^\dagger e^{-\frac{1}{5}\mathrm{tr}B^5 + \mathrm{tr}DD^\dagger B^3 + \mathrm{tr}(DD^\dagger)^2 B + \mathrm{tr}B\Lambda^4} \\ &= \int dB e^{\mathrm{tr}B\Lambda^4 - \frac{1}{5}\mathrm{tr}B^5 - 3k_2\mathrm{tr}\log B - k_2\mathrm{tr}\frac{1}{B^5} - k_2^2\mathrm{tr}\frac{1}{B^2}\mathrm{tr}\frac{1}{B^3} + O\left(\frac{1}{B^{10}}\right)}. \end{aligned} \tag{6.39}$$

JHEP04(2009)110

For general p, after integration over the D-field in a perturbation, we obtain

$$Z = \int dB e^{\mathrm{tr}B\Lambda^p - \frac{1}{p+1}\mathrm{tr}B^{p+1} - (p-1)k_2 \mathrm{trlog}B - \frac{p-2}{2}k_2 \mathrm{tr}\frac{1}{B^{p+1}} - \frac{p-2}{2}k_2^2 \mathrm{tr}\frac{1}{B^2}\mathrm{tr}\frac{1}{B^{p-1}} + \cdots} \tag{6.40}$$

In order to identify the intersection numbers, we expand in powers of $\frac{1}{\Lambda}$. For this purpose, we shift $B \to B + \Lambda$. New terms in the exponent, which are corrections to the logarithmic term, have the form of the product of two traces. In the large Λ case, the fifth term is of the form

$$\mathrm{tr}\frac{1}{(\Lambda+B)^2}\mathrm{tr}\frac{1}{(\Lambda+B)^{p-1}} \sim -2\left(\mathrm{tr}\frac{1}{\Lambda^3}B\right)\cdot \mathrm{tr}\frac{1}{\Lambda^{p-1}} + \cdots \tag{6.41}$$

The term $\mathrm{tr}\frac{1}{\Lambda^{p-1}}$ is t_{p-1}; such terms appear in the $c = 1$ string theory [14, 15].

We now evaluate the intersection numbers for a small number of marked points, and for lower orders in $\frac{1}{\Lambda}$. In this case, we can neglect the above correction terms, and we approximate the partition function by

$$Z = \int dB e^{-\frac{1}{p+1}\mathrm{tr}B^{p+1} - (p-1)k_2 \mathrm{trlog}B + \mathrm{tr}B\Lambda^p} \tag{6.42}$$

We find, at order $\frac{1}{\Lambda^{p+1}}$, up to three marked points,

$$\begin{aligned}\mathrm{log}Z \;=\; & \left(\frac{p-1}{24}\right)\frac{1}{p}\sum\frac{1}{{\lambda_i}^{p+1}} + \left(\frac{p(p-1)}{12}\right)\frac{1}{p}\left(\sum\frac{1}{\lambda_i}\right)^2\left(\sum\frac{1}{{\lambda_i}^{p-1}}\right)\\ & + \left(\frac{p(p-1)}{2}k_2\right)\frac{1}{p}\left(\sum\frac{1}{\lambda_i^2}\right)\left(\sum\frac{1}{{\lambda_i}^{p-1}}\right) + \left(\frac{(p-1)^2}{2}k_2^2\right)\frac{1}{p}\sum\frac{1}{{\lambda_i}^{p+1}}\\ & + (higher\ \ order)\end{aligned} \tag{6.43}$$

where the overall factor $\frac{1}{p}$ is a normalization constant absorbed in λ. Thus we find the two-matrix model for $c = 1$, obtained from the characteristic polynomials, reduces to the one matrix model, and the topological invariants becomes similar to the intersection numbers of p-spin curves.

When we put $p \to -1$, we find that the last term in (6.40) $\left(\mathrm{tr}\frac{1}{B^2}\right)\left(\mathrm{tr}\frac{1}{B^{p-1}}\right)$ behaves like

$$\left(\mathrm{tr}\frac{1}{B^2}\right)\left(\mathrm{tr}\frac{1}{B^{p-1}}\right) = \left(\mathrm{tr}\frac{1}{B^2}\right)(\mathrm{tr}B^2) \sim \left(\mathrm{tr}\frac{1}{\Lambda^2}\right)(\mathrm{tr}B^2) \tag{6.44}$$

where we make a shift $B \to \Lambda + B$. In the case $p \to -1$, when B is order of Λ, all the terms of the potential should be order of one, and indeed $t_m(\mathrm{tr}B^m)$ is order of one for the Λ-dependence. Therefore, the potential has a series of $\sum t_m(\mathrm{tr}B^m)$. Such terms appear in the c=1 string theory from the calculation of the tachyon correlators [14, 15].

7 Discussion

In this article, we have considered the explicit p-dependence of the intersection numbers of moduli spaces of p-th spin curves based on one and two Gaussian matrix models.

In the one-matrix case, the limit $p \to -1$ gives a generating function of the Euler characteristics. In the two-matrix case, we have obtained a logarithmic matrix model with polynomial corrections, which is related to the generating function for the tachyon correlators [14, 15].

The duality, on which the present analysis relies, is the relation between the characteristic polynomials of two different Gaussian matrices. The characteristic polynomials are computed as determinants, expressed in terms of Grassmann variables ψ_i^α,$(i = 1, \ldots, N, \alpha = 1, \ldots k)$. In the large N limit the p-th singularity is tuned through an appropriate choice of the eigenvalues of an external source matrix A. One parameter remains, namely the number of different λ_α, $\alpha = 1 \cdots k$. The Fourier transform with respect to the λ_α yields the correlation function $\mathrm{U}(s_1, \ldots, s_n)$. The symmetry between N and k becomes then implicit. Although this duality might be related to the open/closed string duality [26–28], we have not been able yet to reach a clear picuture in this direction.

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A Three point correlation function $\mathrm{U}(s_1, s_2, s_3)$

For the three-point correlation function $\mathrm{U}(s_1, s_2, s_3)$, we address ourselves to the determinant terms in (2.23) similar to the two-point case. The longest cycle in the determinant of a 3×3 matrix is

$$\det(a_{ij})|_{\text{longest}} = a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \tag{A.1}$$

where $a_{ij} = \frac{1}{u_i - u_j + \frac{1}{2}(s_i + s_j)}$.

We consider the first cycle of (A.1), (the second cycle is almost the same),

$$\begin{aligned}
&\frac{1}{u_1 - u_2 + \frac{1}{2}(s_1 + s_2)} \frac{1}{u_2 - u_3 + \frac{1}{2}(s_2 + s_3)} \frac{1}{u_3 - u_1 + \frac{1}{2}(s_3 + s_1)} \\
&\qquad = \frac{2}{s_1 + s_2 + s_3} \int_0^\infty dx \int_0^\infty dy \mathrm{sh}\left(\frac{x}{2}(s_1 + s_2 + s_3)\right) \\
&\qquad\quad \times \left[e^{-\frac{s_2}{2}x - \frac{s_1+s_2}{2}y - (x+y)u_1 + yu_2 + xu_3} + e^{-\frac{s_2}{2}x - \frac{s_2+s_3}{2}y - xu_1 - yu_2 + (x+y)u_3}\right]
\end{aligned} \tag{A.2}$$

We express the two terms as

$$\mathrm{U}(s_1, s_2, s_3) = U^I + U^{II}. \tag{A.3}$$

JHEP04(2009)110

After the shift $s_i \to \frac{s_i}{N}$, using the notation $N' = N\left(\frac{p-1}{pc}\right)^{\frac{1}{p}}$, we have

$$U^I = \frac{2N'}{s_1+s_2+s_3}\left(\frac{1}{s_3}\right)^{\frac{1}{p}}\int_0^\infty dx\int_0^\infty dy\mathrm{sh}\left(\frac{x}{2N'}s_1^{\frac{1}{p}}(s_1+s_2+s_3)\right)$$
$$e^{-\frac{s_2}{2N'}s_1^{\frac{1}{p}}x-\frac{s_1+s_2}{2N'}s_2^{\frac{1}{p}}y-iv_1\left(x+\left(\frac{s_2}{s_1}\right)^{\frac{1}{p}}y\right)+iyv_2+i\left(\frac{s_1}{s_3}\right)^{\frac{1}{p}}xv_3}G(v_1)G(v_2)G(v_3) \tag{A.4}$$
$$U^{II} = \frac{2N'}{s_1+s_2+s_3}\left(\frac{1}{s_3}\right)^{\frac{1}{p}}\int_0^\infty dx\int_0^\infty dy\mathrm{sh}\left(\frac{x}{2N'}s_1^{\frac{1}{p}}(s_1+s_2+s_3)\right)$$
$$e^{-\frac{s_2}{2N'}s_1^{\frac{1}{p}}x-\frac{s_2+s_3}{2N'}s_2^{\frac{1}{p}}y-iv_1x-iyv_2+i\left(\left(\frac{s_1}{s_3}\right)^{\frac{1}{p}}x+\left(\frac{s_2}{s_3}\right)^{\frac{1}{p}}y\right)v_3}G(v_1)G(v_2)G(v_3) \tag{A.5}$$

where $G(v_i)$ is defined by (4.5). Expanding $G(v_i)$ in powers of $\frac{1}{N'}$, $\mathrm{U}(s_1,s_2,s_3)$ is expressed in terms of the function $\phi_p(x)$.

The intersection numbers $< \tau_{n_1,m_1}\tau_{n_2,m_2}\tau_{n_3,m_3} >$ is obtained from the coefficients of $s_1^{n_1+\frac{m_1+1}{p}}s_2^{n_2+m_2+1p}s_3^{n_3+\frac{m_3+1}{p}}$.

In this three point correlation function, non-trivial genus zero terms appear. From U^{II} in (A.5), we obtain the term $s_1^{\frac{1}{p}}s_2^{\frac{1}{p}}s_3^{1-\frac{1}{p}}$ in the large N' limit. This leads to

$$< \tau_{0,0}\tau_{0,0}\tau_{0,p-2} >_{g=0} = 1 \tag{A.6}$$

Since there is terms of $(\frac{s_2}{s_3})^{\frac{1}{p}}y$ and $(\frac{s_1}{s_3})^{\frac{1}{p}}x$ in (A.5), these terms contribute in the large N limit as $s_1^{\frac{1+q_1}{p}}s_2^{\frac{1+q_2}{p}}s_3^{1-\frac{1+q_1+q_2}{p}}$, and we obtain the intersection numbers,

$$< \tau_{0,q_1}\tau_{0,q_2}\tau_{0,p-2-q_1-q_2} >_{g=0} = 1 \tag{A.7}$$

This is related to the property of ring correlators found in [4]

$$< \tau_{0,q_1}\tau_{0,q_2}\tau_{0,q_3} >_{g=0} = \delta_{q_1+q_2+q_3,p-2} \tag{A.8}$$

which is important for the chiral ring theory and superconformal theory for the primary fields. From this result, the generating function F is obtained as

$$F = \sum < \tau_{0,q_1}\tau_{0,q_2}\tau_{0,q_3} >_{g=0} t_{0,q_1}t_{0,q_2}t_{0,q_3} + O(t^4) \tag{A.9}$$

and the superpotential W can be constructed from the structure constants C_{ijk} defined by

$$C_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} \tag{A.10}$$

where we put

$$t_i = t_{0,i-1}, \qquad (i = 1,\ldots,p-1) \tag{A.11}$$

If we consider only primary field, neglecting gravitational descendants, we only need the terms $\prod_m t_{0,m}$. When we consider this primary field, in the genus zero case, we obtain therefore, for instance for p=5,

$$F = \frac{1}{2}t_{0,0}^2 t_{0,3} + t_{0,0}t_{0,1}t_{0,2} + \frac{1}{3!}t_{0,1}^3 + O(t^4) \tag{A.12}$$

and the structure constants become $C_{114} = C_{123} = C_{222} = 1$ for p=5 case.

B The n-point correlation function for $n \geq 4$

The calculation of the n-point correlation function $\mathrm{U}(s_1, \ldots, s_n)$ at edge singularities follows the same steps as for the n=2 and 3 cases. For the discussion of the higher chiral ring structure, we need more than three points, and thus we consider n≥4.

One of the longest cycle terms in the determinant for the four point correlation function $\mathrm{U}(s_1, s_2, s_3, s_4)$ is

$$
\begin{aligned}
a_{12}a_{23}a_{34}a_{41} &= \frac{1}{s_1+s_2+s_3+s_4}(a_{12}+a_{23})(a_{34}+a_{41}) \\
&\quad \times \left(\frac{1}{u_1-u_3+\frac{1}{2}(s_1+2s_2+s_3)} - \frac{1}{u_1-u_3-\frac{1}{2}(s_1+2s_4+s_3)} \right)
\end{aligned} \tag{B.1}
$$

with

$$
a_{ij} = \frac{1}{u_i-u_j+\frac{1}{2}(s_i+s_j)} \tag{B.2}
$$

This term can be expressed by the integrals,

$$
\begin{aligned}
a_{12}a_{23}a_{34}a_{41} &= -\frac{2}{s_1+s_2+s_3+s_4}\int_0^\infty dxdydz e^{-x(u_1-u_3)-\frac{1}{2}(s_2-s_4)x} \\
&\quad \times \sinh\left(\frac{1}{2}x(s_1+s_2+s_3+s_4)\right) \\
&\quad \times \left[\exp\left(-\frac{1}{2}y(s_1+s_2)-\frac{1}{2}z(s_3+s_4)-u_1y+yu_2-zu_3+zu_4\right)\right. \\
&\quad +\exp\left(-\frac{1}{2}y(s_1+s_2)-\frac{1}{2}z(s_1+s_4)-u_1y+yu_2-zu_4+zu_1\right) \\
&\quad +\exp\left(-\frac{1}{2}y(s_2+s_3)-\frac{1}{2}z(s_3+s_4)-u_2y+yu_3-zu_3+zu_4\right) \\
&\quad \left.+\exp\left(-\frac{1}{2}y(s_2+s_3)-\frac{1}{2}z(s_1+s_4)-u_2y+yu_3-zu_4+zu_1\right)\right]
\end{aligned} \tag{B.3}
$$

Using the same change of variables and scalings as before, we obtain

$$
\mathrm{U}(s_1, s_2, s_3, s_4) = U^I + U^{II} + U^{III} + U^{IV} \tag{B.4}
$$

These four terms are given by $\sigma = s_1+s_2+s_3+s_4$,

$$
\begin{aligned}
U^I &= -\frac{2N'^3}{\sigma}\left(\frac{1}{s_4}\right)^{\frac{1}{p}}\int \frac{dv_i}{(2\pi)^4}\sinh\left(\frac{x}{2N'}s_1^{\frac{1}{p}}\sigma\right)\prod_{i=1}^4 G(v_i) \\
&\quad \times\exp\left[-\frac{1}{2N'}(s_2-s_4)s_1^{\frac{1}{p}}x-\frac{1}{2N'}(s_1+s_2)s_2^{\frac{1}{p}}y-\frac{1}{2N'}(s_3+s_4)s_3^{\frac{1}{p}}z\right. \\
&\quad \left.-ixv_1-i\left(\frac{s_2}{s_1}\right)^{\frac{1}{p}}yv_1+iyv_2+i\left(\frac{s_1}{s_3}\right)^{\frac{1}{p}}xv_3-izv_3+i\left(\frac{s_3}{s_4}\right)^{\frac{1}{p}}zv_4\right]
\end{aligned} \tag{B.5}
$$

JHEP04(2009)110

$$
\begin{aligned}
U^{II} &= -\frac{2N'^3}{\sigma}\left(\frac{1}{s_4}\right)^{\frac{1}{p}}\int\frac{dv_i}{(2\pi)^4}\sinh\left(\frac{x}{2N'}s_1^{\frac{1}{p}}\sigma\right)\prod_{i=1}^{4}G(v_i)\\
&\times\exp\left[-\frac{1}{2N'}(s_2-s_4)s_1^{\frac{1}{p}}x-\frac{1}{2N'}(s_1+s_2)s_2^{\frac{1}{p}}y-\frac{1}{2N'}(s_1+s_4)s_3^{\frac{1}{p}}z\right.\\
&\left.-ixv_1-i\left(\frac{s_2}{s_1}\right)^{\frac{1}{p}}yv_1+i\left(\frac{s_3}{s_1}\right)^{\frac{1}{p}}zv_1+iyv_2+i\left(\frac{s_1}{s_3}\right)^{\frac{1}{p}}xv_3-i\left(\frac{s_3}{s_4}\right)^{\frac{1}{p}}zv_4\right]
\end{aligned}
\tag{B.6}
$$

$$
\begin{aligned}
U^{III} &= -\frac{2N'^3}{\sigma}\left(\frac{1}{s_4}\right)^{\frac{1}{p}}\int\frac{dv_i}{(2\pi)^4}\sinh\left(\frac{x}{2N'}s_1^{\frac{1}{p}}\sigma\right)\prod_{i=1}^{4}G(v_i)\\
&\times\exp\left[-\frac{1}{2N'}(s_2-s_4)s_1^{\frac{1}{p}}x-\frac{1}{2N'}(s_2+s_3)s_2^{\frac{1}{p}}y-\frac{1}{2N'}(s_3+s_4)s_3^{\frac{1}{p}}z\right.\\
&\left.-ixv_1-iyv_2+i\left(\frac{s_1}{s_3}\right)^{\frac{1}{p}}v_3x-izv_3+i\left(\frac{s_2}{s_3}\right)^{\frac{1}{p}}yv_3+i\left(\frac{s_3}{s_4}\right)^{\frac{1}{p}}zv_4\right]
\end{aligned}
\tag{B.7}
$$

$$
\begin{aligned}
U^{IV} &= -\frac{2N'^3}{\sigma}\left(\frac{1}{s_4}\right)^{\frac{1}{p}}\int\frac{dv_i}{(2\pi)^4}\sinh\left(\frac{x}{2N'}s_1^{\frac{1}{p}}\sigma\right)\prod_{i=1}^{4}G(v_i)\\
&\times\exp\left[-\frac{1}{2N'}(s_2-s_4)s_1^{\frac{1}{p}}x-\frac{1}{2N'}(s_2+s_3)s_2^{\frac{1}{p}}y-\frac{1}{2N'}(s_1+s_4)s_3^{\frac{1}{p}}z\right.\\
&\left.-ixv_1+i\left(\frac{s_3}{s_1}\right)^{\frac{1}{p}}zv_1-iyv_2+i\left(\frac{s_1}{s_3}\right)^{\frac{1}{p}}xv_3+i\left(\frac{s_2}{s_3}\right)^{\frac{1}{p}}yv_3-i\left(\frac{s_3}{s_4}\right)^{\frac{1}{p}}zv_4\right]
\end{aligned}
\tag{B.8}
$$

From U^{III}, we obtain the term $s_1^{\frac{2}{p}}s_2^{\frac{2}{p}}s_3^{1-\frac{1}{p}}s_4^{1-\frac{1}{p}}$ which gives the intersection number $<\tau_{0,1}\tau_{0,1}\tau_{0,p-2}\tau_{0,p-2}>_{g=0}$. In this large N' limit, we have

$$
\begin{aligned}
U^{III} &= -\left(\frac{s_1}{s_4}\right)^{\frac{1}{p}}\int_0^\infty dxdydz\int\frac{dv_i}{(2\pi)^4}x\cdot\left(\frac{s_4}{2}s_1^{\frac{1}{p}}x\right)\cdot\left(\frac{1}{2}s_3s_2^{\frac{1}{p}}y\right)\cdot i\left(\frac{s_2}{s_3}\right)^{\frac{1}{p}}yv_3\\
&\exp\left[-\frac{i^p}{p}\sum_i v_i^p-ixv_1+i\left(\frac{s_3}{s_1}\right)^{\frac{1}{p}}zv_1-iyv_2\right]\times\\
&\exp\left[i\left(\frac{s_1}{s_3}\right)xv_3+i\left(\frac{s_2}{s_3}\right)^{\frac{1}{p}}yv_3-i\left(\frac{s_3}{s_4}\right)^{\frac{1}{p}}zv_4\right]
\end{aligned}
\tag{B.9}
$$

Expanding the factors $\exp[i(\frac{s_1}{s_3})xv_3+i(\frac{s_2}{s_3})^{\frac{1}{p}}yv_3-i(\frac{s_3}{s_4})^{\frac{1}{p}}zv_4]$ we obtain the series of the intersection numbers for the primary fields in the genus zero case,

$$
<\tau_{0,q_1}\tau_{0,q_2}\tau_{0,p-q_1-q_2+q_3},\tau_{0,p-2-q_3}>_{g=0}
\tag{B.10}
$$

where $q_1,q_2=1,2,\ldots$, and $q_3=0,1,2,\ldots$. The other three terms U^I,U^{II} and U^{IV} do not yeild terms of the type $s_1^{\frac{q_1+1}{p}}s_2^{\frac{q_2+1}{p}}s_3^{1-\frac{q_1+q_2-q_3}{p}}s_4^{1-\frac{q_3+1}{p}}$.

For p=2, the term (B.10) does not exist, since $\tau_{0,1}$ is not allowed. For higher n-point correlations ($n \geq 5$), there is no correction for the same reason. Therefore, for p=2, the function F for the primary field is

$$F = \frac{1}{6} t_{0,0}^3 \quad (p=2). \tag{B.11}$$

For p=3, we obtain from (B.10) and (A.8),

$$F = \frac{1}{2} t_{0,0}^2 t_{0,1} + \frac{1}{72} t_{0,1}^4 \tag{B.12}$$

For p=4, we obtain (B.10) and (A.8),

$$F = \frac{1}{2} t_{0,0}^2 t_{0,2} + \frac{1}{2} t_{0,0} t_{0,1}^2 + \frac{1}{16} t_{0,1}^2 t_{0,2}^2 + \frac{1}{8 \cdot 5!} t_{0,2}^5 \tag{B.13}$$

The last term is evaluated from the five point correlation function, which has the form, for general p,

$$s_1^{1-\frac{1+q_1}{p}} s_2^{\frac{3+q_2}{p}} s_3^{1-\frac{q_2+1}{p}} s_4^{\frac{3+q_1}{p}} s_5^{1-\frac{1}{p}} \sim t_{0,p-2-q_1} t_{0,2+q_2} t_{0,p-2-q_2} t_{0,2+q_1} t_{0,p-2} \tag{B.14}$$

which leads to the last term in (B.13) for p=4. We have investigated the intersection numbers of primary fields, but other gravity descendants can be obtained in the same ways, which would be in factor of $t_{n,m}$ ($n \neq 0$).

C Ginzburg-Landau potential for primary fields and Gelfand-Dikii equation

The structure constant C_{ijk} defined by (A.10) are obtained from the n-point correlation function through the intersection numbers with n marked points. In this appendix, we discuss the relation to the superpotential [4]. Using the notation

$$t_i = t_{0,i-1} \quad (i = 1, 2, \ldots) \tag{C.1}$$

and the metric $\eta^{nm} = \delta_{n+m,p}$, we define

$$C_{ij}^k = \sum_{m=1}^{p-1} C_{ijm} \eta^{mk}. \tag{C.2}$$

In this notation, F becomes, in the p=4 case for instance,

$$F = \frac{1}{2} t_1^2 t_3 + \frac{1}{2} t_1 t_2^2 + \frac{1}{4} t_2^2 t_3^2 + \frac{1}{60} t_3^5 \tag{C.3}$$

We find that the Witten, Dijkgraaf, Verlinde,@Verlinde relation@[4, 13]

$$C_{ij}^m C_{mkl} = C_{ik}^m C_{mjl} \tag{C.4}$$

holds for the structure constants that we have computed.

JHEP04(2009)110

Then the C_{ij}^k have a ring structure,

$$\phi_i\phi_j = \sum_k C_{ij}^k \phi_k \quad (mod[W'(x)]) \tag{C.5}$$

where ϕ_i is defined by the derivative of the Landau-Ginzburg potential $W(x)$

$$\phi_i = -\frac{\partial W}{\partial t_i} \tag{C.6}$$

We have obtained the function F by the evaluation of the intersection numbers of primary fields up to the 6-point correlation function. For the p=5 case,

$$\begin{aligned} F =& \frac{1}{2}t_{0,0}^2 t_{0,3} + t_{0,0}t_{0,1}t_{0,2} + \frac{1}{6}t_{0,1}^3 + \frac{1}{4}t_{0,1}^2 t_{0,3}^2 \\ &+\frac{1}{2}t_{0,1}t_{0,2}^2 t_{0,3} + \frac{1}{6}t_{0,1}t_{0,2}^3 + \frac{1}{2}t_{0,1}^2 t_{0,2}t_{0,3} \\ &+\frac{1}{12}t_{0,2}^4 + \frac{1}{6}t_{0,2}^2 t_{0,3}^3 + \frac{1}{120}t_{0,3}^6 \end{aligned} \tag{C.7}$$

This leads to $C_{213} = C_{411} = C_{222} = 1, C_{224} = t_4, C_{231} = t_3, C_{232} = t_4, C_{244} = t_2, C_{334} = t_2 + t_4^2, C_{333} = 2t_3, C_{332} = t_4, C_{341} = t_3 t_4, C_{342} = t_3, C_{444} = t_3^2 + t_4^3, C_{434} = 2t_3 t_4$.

The ring structure (C.5) holds with

$$W(x) = \frac{1}{5}x^5 - t_4 x^3 - t_3 x^2 + (t_4^2 - t_2)x + (t_3 t_4 - t_1) \tag{C.8}$$

The function ϕ_i is

$$\phi_i = -\frac{\partial W}{\partial t_i} \tag{C.9}$$

and the equation of the ring structure (C.5) holds with $\mathrm{mod}W'(x) = \mathrm{mod}[x^4 - 3t_3 x^2 - 2t_3 + t_4 - t_2]$.

The Landau-Ginzburg potential W in (C.8) is the same as the superpotential of the twisted N=2 superconformal theory for A_4 type. From the singularity theory, this potential (C.8) is called a swallow tail.

Thus we find that the random matrix theory with an external source for the p-th critical point gives the Landau-Ginzburg potential of the $N = 2$ superconformal theory for the primary fields in the genus zero case. Our integral expression for the n-point correlation function may be used without difficulties to give the intersection numbers and the gravity descendants for higher genus.

We note that these algebraic structures reduce to the Gelfand-Dikii equation, which gives the generalized KdV hierarchies. For instance, in the case p=3, we obtain from our formulation the Boussinesque equation,

$$\frac{\partial^2 F}{\partial t_{0,1}^2} = \frac{\partial^4 F}{\partial t_{0,0}^4} - \frac{2}{3}\left(\frac{\partial^2 F}{\partial t_{0,0}^2}\right)^2 \tag{C.10}$$

and it's higher gravitational desendents. This hierarchy can be derived from Gelfand-Dikii equation [4]. This equation is expressed by [12]

$$i\frac{\partial Q}{\partial t_{n,m}} = \left[Q_+^{n+\frac{m+1}{p}}, Q\right] \cdot \frac{C_{n,m}}{\sqrt{p}} \tag{C.11}$$

which is the generalization of Lax equation. The Q is given

$$Q = D^p - \sum_{i=0}^{p-2} u_i(x) D^i \tag{C.12}$$

and the fraction power of Q is

$$Q^{\frac{1}{p}} = D + \sum_{i>0} w_i D^{-i} \tag{C.13}$$

From this formulation, we obtain the relation to F as

$$\frac{\partial^2 F}{\partial t_{0,0} t_{n,m}} = -C_{n,m} \mathrm{res}\left(Q^{n+\frac{m+1}{p}}\right) \tag{C.14}$$

where

$$C_{n,m} = \frac{(-1)^n p^{n+1}}{(m+1)(p+m+1)\cdots(pn+m+1)} \tag{C.15}$$

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